Current inversion in the Lévy ratchet

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We study the motion of an overdamped test particle in a static periodic potential lacking spatial symmetry under the influence of periodically modulated α-stable (Lévy) type noise. Due to the nonthermal character of the driving noise, the particle exhibits a motion with a preferred direction. The additional periodic modulation of the noise asymmetry changes the behavior of the static "Lévy ratchet." For the fast rate of the noise asymmetry modulation, the Lévy ratchet behaves like the one driven by the symmetric α-stable noise. When the modulation period is larger, the nontrivial effects of the noise asymmetry on the behavior of the Lévy ratchet are visible. In particular, the current inversion is observed in the system at hand. The properties of the Lévy ratchet are studied by use of the robust measures of directionality, which are defined regardless of the type of the stochastic driving.

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I. INTRODUCTION

A ratchet is a noise-assisted device which is able to convert nonequilibrium fluctuations into a directed motion [1–4]. Traditionally, it has been assumed that a ratcheting device consists of the following ingredients: (i) periodic potential with broken spatial symmetry, (ii) thermal noise, and (iii) nonequilibrium perturbation. All these elements are required in order to induce a directed motion [3,4]. However, it was demonstrated that a minimal setup which leads to the ratcheting effect can be built from a smaller number of components than considered so far. The minimal system allowing for occurrence of a directed motion can be built from a periodic potential subjected to the nonequilibrium noise of the α-stable (Lévy) type, which combines the role of the thermal noise and a nonequilibrium fluctuation in the ratcheting device. The α-stable noises are not only of nonequilibrium type [5]. They can also be intrinsically asymmetric. Consequently, it is possible to weaken the assumption about the periodic potential, i.e., it is possible to consider potentials with broken spatial symmetry together with symmetric noises [6,7] or symmetric potentials with asymmetric noises [7].

It has been already shown that the Lévy ratchet driven by the symmetric α-stable noise works in the manner which resembles a thermal ratchet [3], i.e., the directed current is induced into the direction of the steeper slope of the potential [6,7]. The problem of time evolution of the initial probability density has been explored in great detail [6]. It has been shown that a probability density of a particle position attains a power-law, symmetric form with respect to its median, which moves linearly with time [6]. Also the role of the noise asymmetry in systems perturbed by Lévy noises has been studied [8,9]. It has been shown that nonzero asymmetry of the α-stable noise can induce asymmetric stationary densities [10] in single-well symmetric potentials or a current in symmetric periodic potentials [7]. However, the role of the periodic modulation of the noise asymmetry has been studied only in a single-well, symmetric potential [11], where it was shown that the periodic modulation of the noise asymmetry induces the effect of the dynamical hysteresis.

The interplay between asymmetry of the periodic potential and asymmetry of the α-stable noise determines the direction of the current in the Lévy ratchet. Here we continue the line of investigation sketched in [6], i.e., we consider a motion of the Lévy-Brownian particle in the static periodic potential with broken spatial symmetry subjected to the action of the α-stable noise with the periodically modulated asymmetry. The periodic modulation of the noise asymmetry guarantees that in the studied system all perturbations are of "zero averages type." As it will be demonstrated, the periodic modulation of the noise asymmetry increases the richness of the observed phenomena in the minimal Lévy ratchet.

II. MODEL

The studied Lévy ratchet is described by the following overdamped Langevin equation:

$$\frac{dx(t)}{dt} = -V'(x) + \zeta(t), \tag{1}$$

where $V(x)$ is an external static ratchet potential with broken spatial symmetry [3]

$$V(x) = \frac{1}{2\pi}\left[\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)\right]. \tag{2}$$

$\zeta(t)$ stands for the α-stable white noise [12–15]. Initially, at $t=0$, a test particle is located at $x=0$.

$\zeta(t)$ is a nonequilibrium noise which produces stochastic increments Δx which are stationary, independent, and distributed according to the α-stable density [14–16]. The α-stable distributions correspond to a four-parameter family of the unimodal probability density functions [14,15], which are described by the characteristic function

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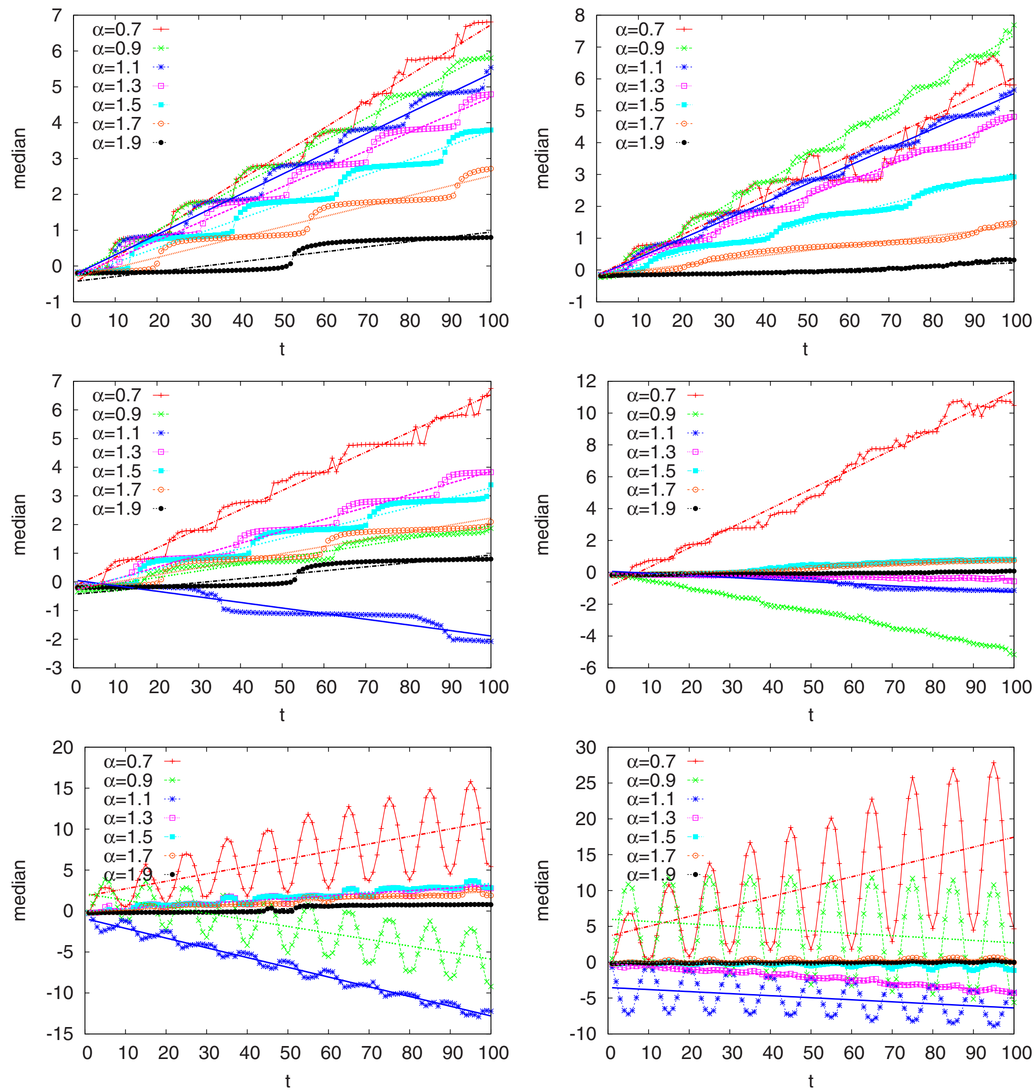

FIG. 1. (Color online) Time dependence of the location of a median which describes the overall motion of the ensemble of particles. Sample trajectories have been numerically estimated from Eq. (1), averaged over 10^6 realizations with time step of integration 10^{-3}, for $\sigma=0.25$ (left column), $\sigma=0.5$ (right column). Various panels correspond to various driving periods: $T_\Omega=0.1$ (top panel), $T_\Omega=1$ (middle panel), and $T_\Omega=10$ (bottom panel).

$$\phi(k) = \exp\left\{ - \sigma^\alpha |k|^\alpha \left[1 - i\beta \,\mathrm{sgn}\, k \tan\left(\frac{\pi\alpha}{2}\right)\right] + ik\mu \right\}. \tag{3}$$

The parameter α $(\alpha \in (0,1) \cup (1,2)]$ is the stability index of the distribution describing its asymptotic power-law tail characteristic, which for large values of argument ζ (for $\alpha < 2$) is of the $\zeta^{-(\alpha+1)}$ type. The parameter β $(\beta \in [-1,1])$ is the asymmetry parameter, i.e., α-stable densities are symmetric for $\beta=0$ otherwise they are intrinsically asymmetric. The parameter μ denotes the location parameter. Finally, the scale parameter σ scales the overall width of the distributions. The Gaussian distribution, see Eq. (3), corresponds to a special case of a Lévy stable law for $\alpha=2$, with μ interpreted as a mean and σ as the dispersion of the distribution. In further studies, we assume $\mu=0$, i.e., we consider strictly α-stable noises only [14,15].

Equation (1) describing the time evolution of the Lévy ratchet is a stochastic differential equation driven by the α-stable noise which calls for special treatment. Consequently, Eq. (1) was approximated by use of appropriate numerical methods [14,15]. Furthermore, as it was shown in [6], standard measures of directionality of the ratchet dynamics [3] are not applicable or are applicable for a limited set of parameters [7]. Therefore in order to quantify the ratcheting effect we use robust measures which were defined and studied in [6].

III. RESULTS

The properties of the Lévy ratchet were studied for the standard ratchet potential given by Eq. (2); see [3]. The value of the noise intensity σ was adjusted to $\sigma=\{0.25, 0.5, 1\}$. The noise asymmetry β was periodically modulated, i.e. $\beta = \sin(\Omega t) = \sin(2\pi t / T_\Omega)$ with the modulation period T_Ω

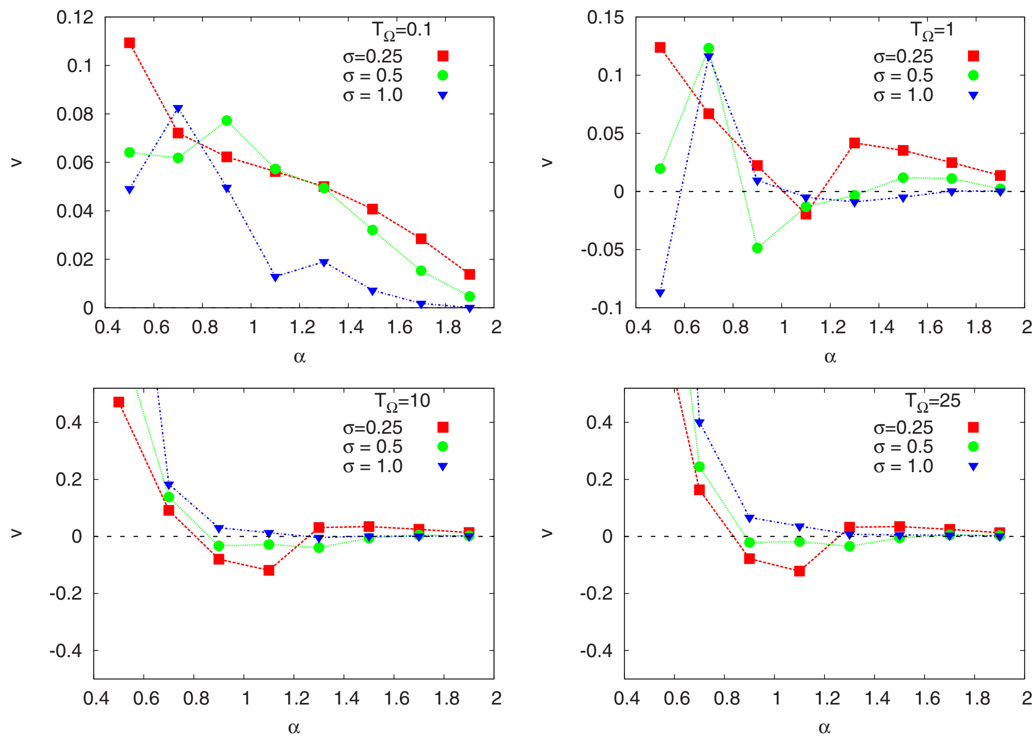

FIG. 2. (Color online) Value of the group velocity v (slope of the median, see Fig. 1) as a function of the stability index α. The lines are drawn to guide the eye. Various panels correspond to different driving periods: $T_\Omega=\{0.1,1,10,25\}$.

$=\{0.1,1,10,25\}$. The properties of the transport in the Lévy ratchet were studied for the whole spectrum of stability indices α (0.5–1.9). The fast modulation of the asymmetry parameter β results in the behavior of the Lévy ratchet very similar to the behavior of the static Lévy ratchet [6]. The slow modulation of the noise asymmetry manifests its presence by producing periodic shape of some measures of the directionality; see Fig. 1.

The dynamics of the Lévy ratchet, see Eq. (1), is studied for the sharp initial condition, i.e., $x(0)=0$. In course of time the width of the probability distribution function (PDF) of the particle positions grows. At the length scales larger than the period of the potential, PDF attains a power-law, symmetric form with respect to its median. The distribution width, characterized by the interquantile distance, grows in time. In the majority of cases this growth is linear. However, there are cases when deviations from a clear linear growth are visible (results not shown).

The location of the median for $T_\Omega=0.1$ (see top panel of Fig. 1) is very similar to the behavior of the median for the static Lévy ratchet [6]. It suggests that, as expected, for the fast rate of the modulation of the asymmetry parameter the system behaves as if it were perturbed by the symmetric α-stable noises. The situation changes when the period of the modulation, T_Ω, is larger. In such a case a particle has more time to adapt to the present value of the noise asymmetry. Consequently, the behavior of the ratchet is influenced by the asymmetry of the driving noise. The first visible consequence, contrary to the static Lévy ratchet [6], is the occurrence of the overall motion into the direction of the smaller slope of the potential; see the middle panel of Fig. 1. A further increase of the period of modulation leads to the addition of the periodic dependence of the median location. In such situations, the overall monotonic trend is decorated with periodic modulations; see the bottom panel of Fig. 1. With the increase of T_Ω the periodic decoration is more apparent.

Figure 2 presents the value of the linear slope fitted to the location of the median characterizing the observed overall trend. Figure 2 again confirms the predictions that for the fast modulation of the asymmetry parameter there is no current inversion in the system at hand. Moreover, values of the group velocity for $T_\Omega=0.1$ are very close to those observed for the static Lévy ratchet [6]. The current inversion takes place for slower rates of noise modulation. It underlines the role of the nonzero noise asymmetry, which is responsible for inducting of the directed motion into the direction of the smaller slope of the potential, i.e., it changes the behavior observed in the static Lévy ratchet [6,7] driven by the symmetric α-stable noise.

For a symmetric noise, the random jumps into the left and right directions are equally probable. For a low noise intensity, a particle lands more often on smaller slopes of the potential and then it continues its motion to the right in the more regular diffusive manner. The movement to the right into the direction of the minima of the potential produces the resultant motion to the right. From a mathematical perspective, it is explained by the possible decomposition of the α-stable process into jump and diffusive parts [17]. The situation changes in the presence of the nonzero temporal noise asymmetry. The periodic modulation of the noise asymmetry breaks the left-right symmetry providing a possibility to induce overall motion in any direction. This occurs under the condition that the rate of the asymmetry modulation is slow enough to give the particle a chance to be moved by the

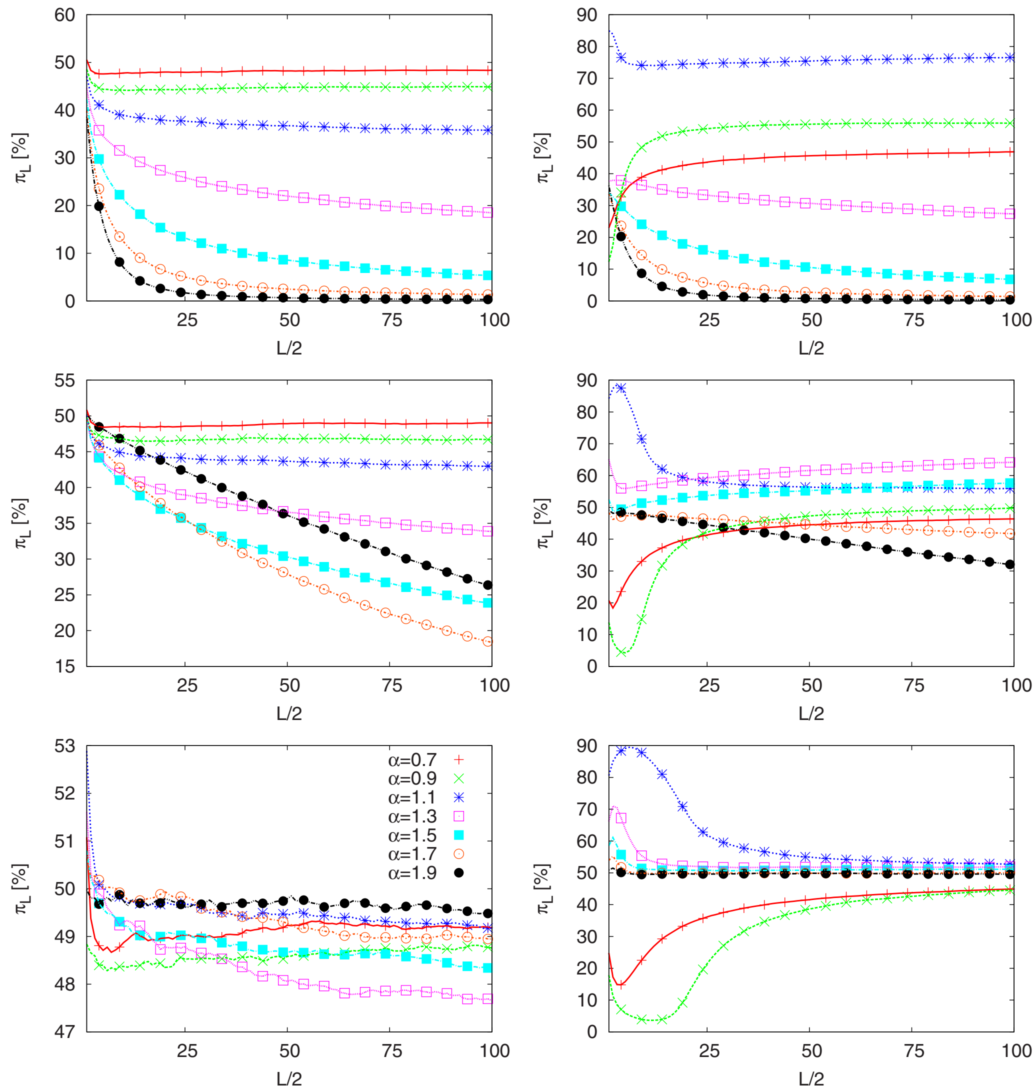

FIG. 3. (Color online) Splitting probability π_L, i.e., probability of the first escape to the left as a function of the box half width $L/2$. Initially a particle is located at $x=0$. Various panels correspond to different values of the scale parameter: $\sigma=0.25$ (top panel), $\sigma=0.5$ (middle panel) and $\sigma=1$ (bottom panel). Various columns correspond to various driving periods: $T_\Omega=0.1$ (left column) and $T_\Omega=10$ (right column).

skewed noise into the direction determined by the noise, not by the potential. Furthermore, with the increasing value of the stability index α, the value of the group velocity decreases. Finally, in the Gaussian limit the directed current is not observed in the system [3].

Figure 3 presents the dependence of the splitting probability π_L as the function of the box half width, $L/2$. The splitting probability π_L is the probability that the particle starting its motion in the middle of the interval of length L performs the first escape from this interval through its left boundary [6]. For the fast rate of the noise modulation, the observed behavior is compatible with the one observed for the static Lévy ratchet [6]. The modulation of the noise asymmetry makes clearer distinction between cases when the distribution of particles' positions moves faster than its width grow, i.e., cases with the value of the stability index α smaller and larger than 1. Furthermore, for the slow rate of the noise modulation, the initial, apparent monotonic behavior of the splitting probability is produced by the noise asymmetry. More precisely, the initial growth of the noise asymmetry moves the particle into one direction, resulting in the values of splitting probability significantly different from 50% for small widths of boxes. After the initial growth, the noise asymmetry starts to decrease and becomes negative. Consequently, a particle is pushed into the opposite direction than initially and the initial monotonic behavior of the splitting probability is reversed. This effect is stronger for larger noise intensities σ, because when the noise intensity is larger, the motion of the particle is less affected by the detailed shape of the deterministic, static potential. With the increasing period of the modulation, the width of the observed peak increases.

The splitting probability manifests behavior compatible with the behavior of the group velocity. In situations when the overall motion is to the right, see Fig. 2, the splitting probability is larger than 50%; see Fig. 3. The excess over 50% is larger for small noise intensities, i.e., in situations when the decomposed diffusive part of the α-stable noise is "more localized." Nevertheless, it is necessary to emphasize the difference between the splitting probability and the group velocity. The group velocity characterizes the overall motion of the probability density in time while the splitting probability describes the preferred direction of the first escape from a finite interval. Consequently, information carried by both measures is not fully equivalent.

IV. SUMMARY AND CONCLUSIONS

The properties of the Lévy ratchet were studied by use of robust measures of the directionality, such as the group velocity and the splitting probability, since these are robust to the dynamics of the studied system and can be applied to the whole spectrum of driving noise parameters. The overall motion of the median of the particle positions distribution defines the group velocity, which is considered as one of the important measures of the current directionality [6].

In the Lévy ratchet driven by symmetric α-stable noises the induced current is observed into the direction of the steeper slope of the potential. This resembles the Lévy ratchet with the so-called thermal ratchet. A nonzero noise asymmetry can destroy this property. For example, it can induce current into the direction of the smaller slope of the potential or induce a directed current in symmetric potentials. The Lévy ratchet driven by nonsymmetric noises is driven by the perturbation which is not of the "zero average type," because the nonzero noise asymmetry β introduces the favored direction. Here, we resolve this delicacy by introducing periodic modulation of the noise asymmetry, which is of the "zero average type." The periodic modulation of the noise asymmetry results in the current inversion in the studied Lévy ratchet.

The main difference between the static Lévy ratchet and the Lévy ratchet driven by the periodically modulated α-stable noise is the possibility of the current inversion. The current inversion takes place for moderate values of the modulation rate of the noise asymmetry. Other characteristics of the evolution of the probability density of the particle position are more robust to the periodic driving. They behave quantitatively in the manner very similar to the one observed in the minimal, static Lévy ratchet [6].

The observed current inversion in the Lévy ratchet is example of the phenomena dynamically induced by the noise. The modulation of the noise asymmetry modifies the character of the interplay between deterministic and stochastic dynamics. The preferred direction of noise pulses periodically changes over time. As a result the particle is more likely to move into the direction of the steeper or smaller slope of the potential. This introduces changes in the overall direction of motion which in turn leads to the current inversion.

The current research represents further development of the theory of stochastic systems driven by the α-stable Lévy noise. It provides a possible extension to earlier applications of the white α-stable noise. The conducted research extends our understanding of the dynamics of systems driven by nonequilibrium noises by investigating in detail the role of the noise asymmetry and its modulation on the behavior of the Lévy ratchet.

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